

# $B_\infty$ -ALGEBRAS, THEIR ENVELOPING ALGEBRAS, AND FINITE SPACES

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**ABSTRACT.** Finite topological spaces are in bijective correspondence with quasi-orders on finite sets. We undertake their study using combinatorial tools that have been developed to investigate general discrete structures.

A particular emphasis will be put on recent topological and combinatorial Hopf algebra techniques. We will show that the set of finite spaces carries naturally generalized Hopf algebraic structures that are closely connected with familiar constructions and structures in topology (such as the one of cogroups in the category of associative algebras that has appeared e.g. in the study of loop spaces of suspensions). The most striking result that we obtain is certainly that the linear span of finite spaces carries the structure of the enveloping algebra of a  $B_\infty$ -algebra.

## 1. INTRODUCTION

Finite topological spaces, or finite spaces, for short, that is, topologies on finite sets, have a long history, going back at least to P.S. Alexandroff [1]. He was the first to investigate, in 1937, finite spaces from a combinatorial point of view and relate them to quasi-ordered sets. Indeed, finite spaces happen to be in bijective correspondence with quasi-orders on finite sets and it is extremely tempting to undertake their study using the combinatorial tools that have been developed to investigate general discrete structures. However, quite surprisingly, such an undertaking does not seem to have taken place so far, and it is the purpose of the present article to do so.

A particular emphasis will be put on recent topological and combinatorial Hopf algebra techniques. We will show that the set of finite spaces carries naturally (generalized) Hopf algebraic structures that are closely connected with usual topological constructions (such as joins or cup products) and familiar structures in topology (such as the one of cogroups in the category of associative algebras, or infinitesimal Hopf algebras, that have appeared e.g. in the study of loop spaces of suspensions and the Bott-Samelson theorem [7, 6]). The most striking result that we obtain is certainly that the linear span of finite spaces carries the structure of the enveloping algebra of a  $B_\infty$ -algebra.

Let us point out that operations such as cup products are usually defined “locally”, that is, inside a chain or cochain algebra associated to a *given* topological space, whereas the structures we introduce hold “globally” over

the linear span of all finite spaces. Although we will not investigate systematically in the present article this interplay between “local” and “global” constructions, it is certainly one of the interesting phenomena showing up in the study of finite topological spaces.

From the historical prospective, a systematic homotopical investigation of finite spaces did not occur till the mid-60’s, with breakthrough contributions by R. E. Stong [27] and M.C. McCord [18, 19]. These investigations were revived in the early 2000s, among others under the influence of P. May; we refer to [2] for details. These studies focussed largely on problems such as reduction methods (methods to remove points from finite spaces without changing their strong or weak homotopy type and related questions such as the construction of minimal spaces, see e.g. [4]), as such they are complementary to the ones undertaken in the present article.

The article is organized as follows: in the next section, we review briefly the links between finite spaces and quasi-orders, introduce the  $Com - As$  structure on finite spaces and study its properties (freeness, involutivity, compatibility with homotopy reduction methods). The third section revisits the equivalent notions of free algebras, cofree coalgebras, cogroups in the category of associative algebras and infinitesimal bialgebras [6, 17, 16]. We extend in particular the results of Livernet and relate these algebras to shuffle bialgebras and their dual bialgebras. Finally, in the last section, we show how these ideas apply to finite spaces, showing in particular that their linear span carries the structure of a cofree coalgebra (in the category of connected coalgebras) and, more precisely, is the enveloping algebra of a  $B_\infty$ -algebra.

In the present article, we study “abstract” finite spaces, that is, finite spaces up to homeomorphisms: we identify two topologies  $\mathcal{T}$  and  $\mathcal{T}'$  on the finite sets  $X$  and  $Y$  if there exists a set map  $f$  from  $X$  to  $Y$  inducing an isomorphism between  $\mathcal{T}$  and  $\mathcal{T}'$ . The study of “decorated” finite spaces (that is, without taken into account this identification) is interesting for other purposes (e.g. enumerative and purely combinatorial ones), it will be the subject of another article.

All vector spaces and algebraic structures (algebras, coalgebras...) are defined over a field  $K$  of arbitrary characteristic. Excepted otherwise stated, the objects we will consider will always be graded and connected (connectedness meaning as usual that the degree 0 component of a graded vector space is the null vector space or the ground field for a graded algebra, coalgebra or bialgebra). Because of this hypothesis, the two notions of Hopf algebras and bialgebras will agree (see e.g. [15]); we will use them equivalently and without further comments.

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## 2. TOPOLOGIES ON FINITE SETS

**2.1. Notations and definitions.** Let  $X$  be a set. Recall that a topology on  $X$  is a family  $\mathcal{T}$  of subsets of  $X$ , called the open sets of  $\mathcal{T}$ , such that:

- (1)  $\emptyset, X \in \mathcal{T}$ .
- (2) The union of an arbitrary number of elements of  $\mathcal{T}$  is in  $\mathcal{T}$ .
- (3) The intersection of a finite number of elements of  $\mathcal{T}$  is in  $\mathcal{T}$ .

When  $X$  is finite, these axioms simplify: a topology on  $X$  is a family of subsets containing the empty set and  $X$  and closed under unions and intersections. In particular, the set of closed sets for  $\mathcal{T}$  (which is automatically closed under unions and intersections) defines a dual topology  $\mathcal{T}^*$  as  $\mathcal{T}^* := \{F \subset X, \exists O \in \mathcal{T}, F = X - O\}$ . We will write sometimes  $\sigma$  for the duality involution,  $\sigma(\mathcal{T}) := \mathcal{T}^*$ ,  $\sigma^2 = Id$ .

Two topologies  $\mathcal{T}, \mathcal{T}'$ , on finite sets  $X$ , resp.  $Y$ , are homeomorphic if and only if there exists a bijective map  $f$  between  $X$  and  $Y$  such that  $f_*(\mathcal{T}) = \mathcal{T}'$  (where we write  $f_*$  for the induced map on subsets of  $X$  and  $Y$ ). We call finite spaces the equivalence classes of finite set topologies under homeomorphisms and write  $\overline{\mathcal{T}}$  for the finite space associated to a given topology  $\mathcal{T}$  on a finite set  $X$ . Every finite space  $\overline{\mathcal{T}}$  can be represented by a (non unique) topology  $\overline{\mathcal{T}}_n$  on a given  $[n] := \{1, \dots, n\}$  (in particular,  $[0] = \emptyset$ ); we call  $\overline{\mathcal{T}}_n$  a standard representation of  $\overline{\mathcal{T}}$ . The duality involution goes over to finite spaces, its action on finite spaces is still written  $\sigma$  (or with a  $*$ ).

Let us recall now the bijective correspondence between topologies on a finite set  $X$  and quasi-orders on  $X$  (see [9]).

- (1) Let  $\mathcal{T}$  be a topology on the finite set  $X$ . The relation  $\leq_{\mathcal{T}}$  on  $X$  is defined by  $i \leq_{\mathcal{T}} j$  if any open set of  $\mathcal{T}$  which contains  $i$  also contains  $j$ . Then  $\leq_{\mathcal{T}}$  is a quasi-order, that is to say a reflexive, transitive relation. Moreover, the open sets of  $\mathcal{T}$  are the ideals of  $\leq_{\mathcal{T}}$ , that is to say the sets  $I \subseteq X$  such that, for all  $i, j \in X$ :

$$(i \in I \text{ and } i \leq_{\mathcal{T}} j) \implies j \in I.$$

- (2) Conversely, if  $\leq$  is a quasi-order on  $X$ , the ideals of  $\leq$  form a topology on  $X$  denoted by  $\mathcal{T}_{\leq}$ . Moreover,  $\leq_{\mathcal{T}_{\leq}} = \leq$ , and  $\mathcal{T}_{\leq_{\mathcal{T}}} = \mathcal{T}$ . Hence, there is a bijection between the set of topologies on  $X$  and the set of quasi-orders on  $X$ . A map between finite topologies (i.e. topologies on finite sets) is continuous if and only if it is quasi-order-preserving.
- (3) Let us define for each point  $x \in X$  the set  $U_x$  to be the minimal open set containing  $x$ . The  $U_x$  form a basis for the topology of  $X$  called the minimal basis of  $\mathcal{T}$ . The quasi-order that has just been introduced can be equivalently defined by  $x \leq_{\mathcal{T}} y \iff y \in U_x$ . Notice that the opposite convention (defining a quasi-order from a topology using the requirement  $x \in U_y$ ) would lead to equivalent results.
- (4) Let  $\mathcal{T}$  be a topology on  $X$ . The relation  $\sim_{\mathcal{T}}$  on  $X$ , defined by  $i \sim_{\mathcal{T}} j$  if  $i \leq_{\mathcal{T}} j$  and  $j \leq_{\mathcal{T}} i$ , is an equivalence on  $X$ . Moreover, the set  $X / \sim_{\mathcal{T}}$  is partially ordered by the relation defined on the equivalence classes

$\bar{i}$  by  $\bar{i} \leq_{\mathcal{T}} \bar{j}$  if  $i \leq j$ . Consequently, we shall represent quasi-orders on  $X$  (hence, topologies on  $X$ ) by the Hasse diagram of  $X/\sim_{\mathcal{T}}$ , the vertices being the equivalence classes of  $\sim_{\mathcal{T}}$ .

- (5) Duality between topologies is reflected by the usual duality of quasi-orders:  $i \leq_{\mathcal{T}^*} j \Leftrightarrow j \leq_{\mathcal{T}} i$ . In particular, the Hasse diagram of  $\mathcal{T}^*$  is obtained by reversing (upside-down) the Hasse diagram of  $\mathcal{T}$ .
- (6) A topological space is  $T_0$  if it satisfies the separation axiom according to which the relation  $\sim$  is trivial (equivalence classes for  $\sim$  are singletons, that is, for any two points  $x, y \in X$ , there always exist an open set containing only one of them). At the level of  $\leq_{\mathcal{T}}$  this amounts to require the antisymmetry: the quasi-order  $\leq_{\mathcal{T}}$  is then a partial order. In other terms, finite  $T_0$ -spaces are in bijection with isomorphism classes of finite partially ordered sets (posets).

For example, here are the topologies on  $[n]$ ,  $n \leq 3$ :

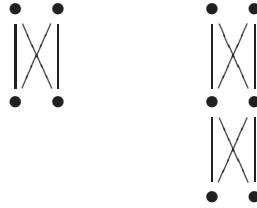
$$\begin{aligned} 1 &= \emptyset; \bullet_1; \bullet_1 \bullet_2, \mathbf{1}_1^2, \mathbf{1}_2^1, \bullet_1 \bullet_2; \\ &\bullet_1 \bullet_2 \bullet_3, \mathbf{1}_1^2 \bullet_3, \mathbf{1}_1^3 \bullet_2, \mathbf{1}_2^1 \bullet_3, \mathbf{1}_2^3 \bullet_1, \mathbf{1}_3^1 \bullet_2, \\ &\mathbf{1}_3^2 \bullet_1, {}^2\mathbf{V}_1^3, {}^1\mathbf{V}_2^3, {}^1\mathbf{V}_3^2, {}^2\mathbf{\Lambda}_{3,1}^1, {}^1\mathbf{\Lambda}_{3,1}^2, {}^1\mathbf{\Lambda}_2^3, \mathbf{1}_1^3, \mathbf{1}_2^2, \mathbf{1}_2^3, \mathbf{1}_2^1, \mathbf{1}_3^1, \mathbf{1}_3^2, \\ &\bullet_1 \bullet_2 \bullet_3, \bullet_1 \bullet_3 \bullet_2, \bullet_2 \bullet_3 \bullet_1, \mathbf{1}_{1,2}^3, \mathbf{1}_{1,3}^2, \mathbf{1}_{2,3}^1, \mathbf{1}_3^{1,2}, \mathbf{1}_2^{1,3}, \mathbf{1}_1^{2,3}, \bullet_1 \bullet_2 \bullet_3. \end{aligned}$$

The two topologies on  $[3]$ ,  ${}^2\mathbf{V}_1^3$  and  ${}^2\mathbf{\Lambda}_{3,1}^1$ , are dual.

A finite space will be represented by an unlabelled Hasse diagram. The cardinalities of the equivalence classes of  $\sim_{\mathcal{T}}$  are indicated on the diagram associated to  $\mathcal{T}$  if they are not equal to 1. Here are the finite spaces of cardinality  $\leq 3$ :

$$1 = \emptyset; \bullet; \dots, \mathbf{1}, \bullet_2; \dots, \mathbf{1}, \mathbf{V}, \mathbf{\Lambda}, \mathbf{1}, \bullet_2, \mathbf{1}_2, \mathbf{1}^2, \bullet_3.$$

The (minimal) finite space realization of the circle and of the 2-dimensional sphere (see e.g. [3])



are examples of self-dual finite spaces.

The number  $t_n$  of topologies on  $[n]$  is given by the sequence A000798 in [25]:

$n$	1	2	3	4	5	6	7	8	9	10
$t_n$	1	4	29	355	6 942	209 527	9 535 241	642 779 354	63 260 289 423	8 977 053 873 043

The set of topologies on  $[n]$  will be denoted by  $\mathbf{T}_n$ , and we put  $\mathbf{T} = \bigsqcup_{n \geq 0} \mathbf{T}_n$ .

The number  $f_n$  of finite spaces with  $n$  elements is given by the sequence A001930 in [25]:

$n$	1	2	3	4	5	6	7	8	9	10
$t_n$	1	3	9	33	139	718	4 535	35 979	363 083	4 717 687

The set of finite spaces with  $n$  elements will be denoted by  $\mathbf{F}_n$ , and we put  $\mathbf{F} = \bigsqcup_{n \geq 0} \mathbf{F}_n$ . The linear span of all finite spaces is written  $\mathcal{F}$  and its (finite dimensional) degree  $n$  component, the linear span of finite spaces with  $n$  elements,  $\mathcal{F}_n$ . We will be from now on interested in the fine structure of  $\mathcal{F}$  in relation to classical topological properties and constructions.

**2.2. Homotopy types.** The present section and the following survey the links between finite spaces and topological notions such as homotopy types. We refer to Stong's seminal paper [27] and to Barmak's thesis [2] on which this account is based for further details and references.

For a finite space, the three notions of connectedness, path-connectedness and order-connectedness agree (the later being understood as connectedness of the graph of the associated quasi-order).

For  $f, g$  continuous maps between the finite spaces  $X$  and  $Y$ , we set

$$f \leq g \Leftrightarrow \forall x \in X, f(x) \leq g(x).$$

This quasi-order on the (finite) mapping space  $Y^X$  is the one associated to the compact-open topology. It follows immediately, among others, that two comparable maps are homotopic and that a space with a maximal or minimal element is contractible (since the constant map to this point will be homotopic to any other map—in particular the identity map).

For the same reason, given a finite space  $X$ , there exists a homotopy equivalent finite space  $X_0$  which is  $T_0$  (the quotient space  $X / \sim_{\mathcal{T}}$  considered in the previous section, for example). Therefore, since [1], the study of homotopy types of finite spaces is in general restricted to  $T_0$  spaces. Characterizing homotopies (inside the category of finite spaces) is also a simple task: two maps  $f$  and  $g$  are homotopic if and only if there exists a sequence:

$$f = f_0 \leq f_1 \geq f_2 \leq \dots \geq f_n = g.$$

In the framework of finite spaces, a reduction method refers to a combinatorial method allowing to remove points from a finite space without changing given topological properties (such as the homotopy type). Stong's reduction method allows a simple and effective construction of representatives of finite homotopy types [27]. Stong first defines the notions of linear and colinear points (also called up beat points and down beat points in a later terminology): a point  $x \in X$  is linear if  $\exists y \in X, y > x$  and  $\forall z > x, z \geq y$ . Similarly,  $x \in X$  is colinear if  $\exists y \in X, y < x$  and  $\forall z < x, z \leq y$ . It follows from the combinatorial characterization of homotopies that, if  $x$  is a linear or colinear point in  $X$ , then  $X$  is homotopy equivalent to  $X - \{x\}$ .

Together with the fact that any finite space is homotopy equivalent to a  $T_0$  space, the characterization of homotopy types follows. A space is called a core (or minimal finite space) if it has no linear or colinear points. By reduction to a  $T_0$  space and recursive elimination of linear and colinear points, any finite space  $X$  is homotopy

equivalent to a core  $X_c$  that can be shown to be unique (recall that we consider finite spaces up to homeomorphism) [27, Thm. 4].

**2.3. Simplicial realizations.** Another important tool to investigate topologically finite spaces is through their connection with simplicial complexes. We survey briefly the results of McCord, following [19, 2].

Recall that a weak homotopy equivalence between two topological spaces  $X$  and  $Y$  is a continuous map  $f : X \rightarrow Y$  such that for all  $x \in X$  and all  $i \geq 0$ , the induced map  $f_* : \pi_i(X, x) \rightarrow \pi_i(Y, f(x))$  is an isomorphism (of groups for  $i > 0$ ). The finiteness requirement enforces specific properties of finite spaces: for example, contrary to what happens for CW-complexes (Whitehead's theorem), there are weakly homotopy equivalent finite spaces with different homotopy types.

The key to McCord's theory is the definition of functors between the categories of finite spaces and simplicial complexes (essentially the categorical nerve and the topological realization). Concretely, to a finite space  $X$  is associated the simplicial complex  $\mathcal{K}(X)$  of non empty chains of  $X/\sim_{\mathcal{T}}$  (that is, sequences  $x_1 < \dots < x_n$  in  $X/\sim_{\mathcal{T}}$ ). Conversely, to the simplicial complex  $\mathcal{K}(X)$  is associated its topological realization  $|\mathcal{K}(X)|$ : the points  $x$  of  $|\mathcal{K}(X)|$  are the linear combinations  $t_1x_1 + \dots + t_nx_n$ ,  $\sum_{i=1}^n t_i = 1$ ,  $t_i > 0$ . Setting  $Sup(x) := x_1$ , McCord's fundamental theorem states that:

$$Sup : |\mathcal{K}(X)| \rightarrow X/\sim_{\mathcal{T}}$$

is a weak homotopy equivalence. In particular,  $|\mathcal{K}(X)|$  is weakly homotopy equivalent to  $X$ . Notice also that  $\mathcal{K}(X)$  and  $\mathcal{K}(X^*)$ , resp.  $|\mathcal{K}(X)|$  and  $|\mathcal{K}(X^*)|$  are canonically isomorphic: a finite space is always weakly homotopy equivalent to its dual.

### 3. SUMS AND JOINS

We investigate from now on operations on finite spaces. Besides their intrinsic interest and their connexions to various classical topological constructions, they are meaningful for the problem of enumerating finite spaces (see e.g. [26, 24, 9]). They will also later underly the construction of  $B_{\infty}$ -algebra structures.

**Notations.** Let  $O \subseteq \mathbb{N}$  and let  $n \in \mathbb{N}$ . The set  $O(+n)$  is the set  $\{k+n \mid k \in O\}$ .

**Definition 1.** Let  $\mathcal{T} \in \mathbf{T}_n$  and  $\mathcal{T}' \in \mathbf{T}_{n'}$  be standard representatives of  $\overline{\mathcal{T}} \in \mathbf{F}_n$  and  $\overline{\mathcal{T}'} \in \mathbf{F}_{n'}$ .

- (1) The topology  $\mathcal{T}.\mathcal{T}'$  is the topology on  $[n+n']$  which open sets are the sets  $O \sqcup O'(+n)$ , with  $O \in \mathcal{T}$  and  $O' \in \mathcal{T}'$ . The finite space  $\overline{\mathcal{T}.\mathcal{T}'}$  is  $\overline{\mathcal{T}}.\overline{\mathcal{T}'}$ .
- (2) The topology  $\mathcal{T} \succ \mathcal{T}'$  is the topology on  $[n+n']$  which open sets are the sets  $O \sqcup [n'](+n)$ , with  $O \in \mathcal{T}$ , and  $O'(+n)$ , with  $O' \in \mathcal{T}'$ . The finite space  $\overline{\mathcal{T} \succ \mathcal{T}'}$  is  $\overline{\mathcal{T}} \succ \overline{\mathcal{T}'}$ .

We omit the proof that the products  $\overline{\mathcal{T}.\mathcal{T}'}$  and  $\overline{\mathcal{T} \succ \mathcal{T}'}$  are well-defined and do not depend on the choice of a standard representative.

The first product is the sum (coproduct, disjoint union) of topological spaces.

The second one deserves to be called the join. Recall indeed that the join  $A * B$  of two topological spaces  $A$  and  $B$  is the quotient of  $[0, 1] \times A \times B$  by the relations  $(0, a, b) \sim (0, a, b')$  and  $(1, a, b) \sim (1, a', b)$ . For example, the join of the  $n$  and  $m$

dimensional spheres is the  $n + m + 1$ -dimensional sphere. When it is defined that way, the join is not an internal operation on finite spaces. However, recall that the join of two simplicial complexes  $K$  and  $L$  is the simplicial complex  $K * L := K \amalg L \amalg \{\sigma \cup \beta, \sigma \in K, \beta \in L\}$  and that the join operation commutes to topological realizations in the sense that (up to canonical isomorphisms)  $|K * L| = |K| * |L|$ . It follows therefore from McCord's theory that, up to a weak homotopy equivalence, the product  $\succ$  is nothing but (a finite spaces version of) the topological join.

**Examples.**

$${}^2\mathbb{V}_1^3 \cdot \mathbb{I}_2^1 = {}^2\mathbb{V}_1^3 \cdot \mathbb{I}_5^4, \quad {}^2\mathbb{V}_1^3 \succ \mathbb{I}_2^1 = {}^2\mathbb{V}_1^3 \cdot \mathbb{I}_5^4.$$

The join of two circles (see above the minimal finite space representation of a circle) is a 3-sphere:



**Proposition 2.** *These two products are associative, with  $\emptyset = 1$  as a common unit. The first product is also commutative. They are compatible with the duality involution:*

$$X^* \cdot Y^* = (X \cdot Y)^*, \quad Y^* \succ X^* = (X \succ Y)^*.$$

The proof is left to the reader.

**Definition 3.** *We extend the two products defined earlier to  $\mathbf{F}$ . Let  $X \in \mathbf{F}$ , different from 1. Notice that  $X$  is connected if and only if it cannot be written in the form  $X = X' \cdot X''$ , with  $X', X'' \neq 1$ .*

- (1) *We shall say that  $X$  is join-indecomposable if it cannot be written in the form  $X = X' \succ X''$ , with  $X', X'' \neq 1$ .*
- (2) *We shall say that  $X$  is irreducible if it is both join-indecomposable and connected.*

The triple  $(\mathbf{F}, \cdot, \succ)$  is a *Com-As* algebra, that is an algebra with a first commutative and associative product and a second, associative, product sharing the same unit. This is a particular example of a 2-associative algebra [17], that is to say an algebra with two associative products sharing the same unit.

For further use, notice the important property that the join-product of two non empty spaces is a connected space (from now on, unless otherwise stated, space means finite space).

**Proposition 4.** (1) *The commutative algebra  $(\mathbf{F}, \cdot)$  is freely generated by the set of connected spaces.*

- (2) *The associative algebra  $(\mathbf{F}, \succ)$  is freely generated by the set of join-indecomposable spaces.*

- (3) *The Com – As algebra  $(\mathbf{F}, \cdot, \succ)$  is freely generated by the set of irreducible spaces.*

*Proof.* 1. Any space can be written uniquely as a sum of connected spaces.

2. Notice first that  $X = Y \succ Z$  if and only if  $Y <_{\mathcal{T}} Z$  (in the sense that, for arbitrary  $y \in Y$ ,  $z \in Z$ ,  $y <_T z$ ). That is,

$$X = Y \succ Z \Leftrightarrow X = Y \coprod Z \text{ and } Y <_{\mathcal{T}} Z.$$

Let us assume that  $X = X_1 \succ X_2 \succ \dots \succ X_n = Y_1 \succ Y_2 \succ \dots \succ Y_m$  with the  $X_i$  and the  $Y_j$  join-indecomposable. Then,  $X_1 \cap Y_1$  is not empty (this would imply for example that  $Y_1 \subset X_2 \succ \dots \succ X_n >_{\mathcal{T}} X_1$ , and similarly  $X_1 >_{\mathcal{T}} Y_1$ , which leads immediately to a contradiction). Moreover,  $X_1 \cap Y_1 <_T X_1 \cap (Y_2 \succ \dots \succ Y_m)$ . A contradiction follows if  $X_1 \cap Y_1 \neq X_1, Y_1$  since we would then have

$$X_1 = (X_1 \cap Y_1) \succ (X_1 \cap (Y_2 \succ \dots \succ Y_m)).$$

We get  $X_1 = Y_1$  and  $X_2 \succ \dots \succ X_n = Y_2 \succ \dots \succ Y_m$ , and the statement follows by induction.

3. Let us describe briefly the free Com – As algebra  $CA(S)$  over a set  $S$  of generators (we write  $\cdot$  and  $\succ$  for the two products). A  $\mathbb{N}^*$ -graded basis  $B = S \coprod B_C \coprod B_A$  of  $CA(S)$  with  $B_1 = S$ ,  $B_C = \coprod_{n \geq 2} B_{C,n}$ ,  $B_A = \coprod_{n \geq 2} B_{A,n}$  can be constructed recursively as follows (in the following  $B_{A,1} = B_{C,1} := B_1 = S$  and the  $\cdot$  product is commutative so that  $a \cdot b = b \cdot a$ ):

$$\begin{aligned} \bullet B_{C,n} &:= \coprod_{n_1 + \dots + n_k = n} \{a_1 \cdot \dots \cdot a_k, a_i \in B_{A,n_i}\} \\ \bullet B_{A,n} &:= \coprod_{n_1 + \dots + n_k = n} \{a_1 \succ \dots \succ a_k, a_i \in B_{C,n_i}\}. \end{aligned}$$

Now, let  $X$  be a space, then one and only one of the three following cases holds

- (1) Either  $X$  is irreducible.
- (2) Either  $X$  is connected but not irreducible, and then it decomposes uniquely into a product  $X = X_1 \succ \dots \succ X_k$  of join-indecomposable spaces.
- (3) Either  $X$  is not connected, and then it decomposes uniquely into a sum  $X = X_1 \cup \dots \cup X_k$  of connected spaces.

It follows by induction that the set of spaces identifies with the basis of the free Com – As algebra over irreducible spaces: writing  $S$  for the latter set, the first case in the previous list corresponds to the case  $X \in S$ ; the second to  $X \in B_A$  with the  $X_i$  in  $B_C$  or  $S$ ; the third to  $X \in B_C$  with the  $X_i$  in  $B_A$  or  $S$ .  $\square$

#### 4. $B_{\infty}$ -ALGEBRAS AND TENSOR ALGEBRAS

The notion of  $B_{\infty}$ -algebra was introduced by Getzler-Jones in the category of chain complexes [14], we consider here the simpler notion of  $B_{\infty}$ -algebra in the subcategory of connected graded vector spaces following e.g. [17]. Concretely, let  $V$  be a graded and connected ( $V_0 = 0$ ) vector space and  $T(V)$  the tensor algebra  $T(V) := \bigoplus_{n \in \mathbb{N}} V^{\otimes n}$  over  $V$  equipped with the deconcatenation coproduct  $\Delta$ , so that

$$\Delta(v_1 \dots v_n) := \sum_{i=0}^n v_1 \dots v_i \otimes v_{i+1} \dots v_n,$$



and  $(T(V), \Delta)$  is the cofree coalgebra over  $V$  (in the category of connected coalgebras : the general structure of cofree coalgebras is more subtle, see [15]). Notice that we use the shortcut notation  $v_1 \dots v_n$  for  $v_1 \otimes \dots \otimes v_n \in V^{\otimes n}$ .

A  $B_\infty$ -algebra structure on  $V$  is, by definition, a Hopf algebra structure on  $T(V)$  equipped with the deconcatenation coproduct. That is, an associative algebra structure on  $T(V)$  compatible with the cofree coalgebra structure on  $T(V)$  (i.e. such that the product is a coalgebra map) [14, p. 48]. Since  $T(V)$  is cofree as a coalgebra, the product map from  $T(V) \otimes T(V)$  to  $T(V)$  is entirely characterized by its image on the subspace  $V$ . This yields to another, equivalent, but less tractable and transparent, definition, of  $B_\infty$ -algebras in terms of structure maps  $M_{p,q} : V^{\otimes p} \otimes V^{\otimes q} \mapsto V$ ,  $p, q \geq 0$  satisfying certain compatibility relation that can be deduced from the associativity of the product –we refer again to [14] for details.

There is in particular an obvious equivalence of categories between the category  $\mathcal{B}_\infty$  of  $B_\infty$ -algebras and the category  $\mathcal{H}_{cof}$  of Hopf algebras that are cofree as connected coalgebras (cofree Hopf algebras, for short). The corresponding functor from  $\mathcal{H}_{cof}$  to  $\mathcal{B}_\infty$  is the functor  $Prim$  of primitive elements (for  $H$  a Hopf algebra,  $Prim(H) := \{h \in H, \Delta(h) = h \otimes 1 + 1 \otimes h\}$ ). This is because, for a cofree coalgebra  $T(V)$ ,  $Prim(T(V)) = V$  –this follows immediately from the definition of the deconcatenation coproduct. For consistency, morphisms between cofree Hopf algebras  $H$  and  $H'$  in  $\mathcal{H}_{cof}$  are required to be induced as coalgebra maps by maps between  $Prim(H)$  and  $Prim(H')$ . The opposite functor  $\mathcal{U}$  from  $\mathcal{B}_\infty$  to  $\mathcal{H}_{cof}$  is given by  $\mathcal{U}(V) := T(V)$ . By analogy with the usual equivalence of categories between graded connected cocommutative Hopf algebras and graded connected Lie algebras (also obtained through the  $Prim$  functor), it is natural to call  $T(V)$ , for  $V$  a  $B_\infty$ -algebra, the  $B_\infty$ -enveloping algebra of  $V$ .

There are various ways to give an algebraic and combinatorial characterization of  $B_\infty$ -structures and cofree Hopf algebras, following ideas that are scattered in the litterature and seem to originate in the Bott-Samelson theorem, according to which  $H_*(\Omega \Sigma X; K) = T(H_*(X; K))$ , where  $\Sigma$  is the suspension functor acting on topological spaces and  $\Omega$  the loop space functor, and in the work of Baues on the bar/cobar construction [5], [14, p. 48]. The paper [17] addresses the problem explicitly, but other approaches follow from [6, 12, 21, 16], and no unified treatment seems to have been given up to date. We take therefore the opportunity of the present article and the existence of  $B_\infty$ -structures on finite spaces (to be introduced in the next section) to present such a short and self-contained treatment. In the process, we extend the results of Livernet [16] on cogroups and infinitesimal bialgebras.

Recall first some generalities on the tensor algebra  $H := T(V)$ . It carries two products (concatenation, shuffle) and two coproducts (deconcatenation, unshuffling coproduct dual to the shuffle product), see [22]. These algebra/coalgebra structures pave the way to various abstract characterizations of tensor algebras.

The first one, historically, is due to Bernstein [6], whose work was influential in the late 90's, when the theory of operads enjoyed a revival after the seminal works of Getzler, Jones, Kapranov, Kontsevich, and others on Koszul duality and algebras up to homotopy. We refer in particular to the works on cogroups and comonoids in categories of algebras over operads [13, 12, 21, 20] to which the forthcoming developments are closely related (although we will focus on the cocommutative case, whereas these articles address the structure of arbitrary cogroups).

The coproduct  $*$  in the category  $\mathcal{A}s$  of connected graded associative algebras, or free product, is obtained as follows: let  $H_1 = K \oplus \overline{H}_1$ ,  $H_2 = K \oplus \overline{H}_2$  be two such algebras, then:

$$H_1 * H_2 := K \oplus \bigoplus_{n \in \mathbb{N}^*} (H_1 * H_2)^{(n)} := K \oplus \bigoplus_{n \in \mathbb{N}^*} [(1, H^{\otimes n}) \oplus (2, H^{\otimes n})],$$

where  $(1, H^{\otimes n})$  (resp.  $(2, H^{\otimes n})$ ) denotes alternating tensor products of  $\overline{H}_1$  and  $\overline{H}_2$  of length  $n$  starting with  $\overline{H}_1$  (resp.  $\overline{H}_2$ ). For example,  $(2, H^{\otimes 4}) = \overline{H}_2 \otimes \overline{H}_1 \otimes \overline{H}_2 \otimes \overline{H}_1$ . The product of two tensors  $h_1 \otimes \dots \otimes h_n$  and  $h'_1 \otimes \dots \otimes h'_m$  in  $H_1 * H_2$  is defined as the concatenation product  $h_1 \otimes \dots \otimes h_n \otimes h'_1 \otimes \dots \otimes h'_m$  when  $h_n$  and  $h'_1$  belong respectively to  $\overline{H}_1$  and  $\overline{H}_2$  (or to  $\overline{H}_2$  and  $\overline{H}_1$ ), and else as:  $h_1 \otimes \dots \otimes (h_n \cdot h'_1) \otimes \dots \otimes h'_m$ .

When  $H_1 = T(V_1)$  and  $H_2 = T(V_2)$ , one gets  $H_1 * H_2 = T(V_1 \oplus V_2)$ . Moreover, by universal properties of free algebras, the linear map  $\iota$  from  $V$  to  $T(V) * T(V)$  defined by

$$(1) \quad \iota(v) := (1, v) + (2, v)$$

induces an algebra map from  $T(V)$  to  $T(V) * T(V)$  which is associative, unital ( $\iota(x) = (1, x) + (2, x) + z$  with  $z \in \bigoplus_{n \geq 2} (H_1 * H_2)^{(n)}$ ) and cocommutative. Equivalently,  $T(V)$  is a cocommutative cogroup in  $\mathcal{A}s$ . Bernstein's fundamental result in view of our forthcoming developments is that any such cogroup is actually naturally isomorphic to a  $T(V)$  [6, Cor. 2.6].

In general, the structure map  $\phi : H \rightarrow H * H$  of a cocommutative cogroup in  $\mathcal{A}s$  is entirely determined by its restriction  $\overline{\Delta}$  on the image to the component  $(1, H \otimes H) \cong \overline{H} \otimes \overline{H}$  of  $H * H$ . Namely,

$$(2) \quad \phi(a) = \sum_{n \geq 1} (1, \overline{\Delta}^{[n-1]}(a)) + (2, \overline{\Delta}^{[n-1]}(a)),$$

where  $\overline{\Delta}^{[n-1]}$  stands for the iterated (coassociative) coproduct from  $\overline{H}$  to  $\overline{H}^{\otimes n}$ . Using the notation  $\overline{\Delta}(x) = \overline{x}_1 \otimes \overline{x}_2$  (and more generally  $\overline{\Delta}^{[n-1]}(x) = \overline{x}_1 \otimes \dots \otimes \overline{x}_n$ ), the coproduct  $\overline{\Delta}$  satisfies the identity

$$(3) \quad \overline{\Delta}(x \cdot y) = x \otimes y + x \cdot \overline{y}_1 \otimes \overline{y}_2 + \overline{x}_1 \otimes \overline{x}_2 \cdot y$$

so that, for  $\Delta(x) := \overline{\Delta}(x) + x \otimes 1 + 1 \otimes x$ , with the notation  $\Delta(x) = x_1 \otimes x_2$  we get the identity

$$(4) \quad \Delta(x \cdot y) = x \cdot y_1 \otimes \overline{y}_2 + x_1 \otimes x_2 \cdot y - x \otimes y$$

that defines on the associative algebra  $H$  equipped with the coproduct  $\Delta$  the structure of an infinitesimal bialgebra.

Conversely, this identity (3) is enough to ensure that

$$\begin{aligned} \overline{\Delta}^{[k]}(x \cdot y) &= \sum_{i=1}^k x_1 \otimes \dots \otimes x_i \otimes y_1 \otimes \dots \otimes y_{k+1-i} \\ &+ \sum_{i=1}^{k+1} x_1 \otimes \dots \otimes x_i \cdot y_1 \otimes \dots \otimes y_{k+2-i}, \end{aligned}$$

from which it follows that  $\phi$ , as defined by the equation (2) defines a cogroup structure on  $H$ . We refer to Livernet [16], to whom these results (the isomorphism of

categories between infinitesimal bialgebras and cogebras in the category of connected graded associative algebras) are due, for further details.

Let us go now one step further and investigate tensor algebras from the point of view of shuffle products and unshuffling coproducts (also called Zinbiel and coZinbiel products/coproducts in the literature, we stick to the classical terminology). Recall first from [23] that the shuffle product  $\sqcup$  is characterized abstractly by the identity involving the left and right half-shuffles  $\prec, \succ$  ( $\sqcup = \prec + \succ$ ):

$$(5) \quad x \prec y = y \succ x, \quad (x \prec y) \prec z = x \prec (y \prec z + y \succ z),$$

where  $\prec, \succ$  are defined recursively on  $T(V)$  by the identities

$$\begin{aligned} x_1 \prec y_1 &:= x_1 y_1, & x_1 \dots x_n \prec y_1 \dots y_m &:= x_1 (x_2 \dots x_n \sqcup y_1 \dots y_m), \\ x_1 \succ y_1 &:= y_1 x_1, & x_1 \dots x_n \succ y_1 \dots y_m &:= y_1 (x_1 \dots x_n \sqcup y_2 \dots y_m). \end{aligned}$$

A shuffle bialgebra is a commutative Hopf algebra whose product, written  $\sqcup$  is a shuffle product (that is, can be written  $\sqcup = \prec + \succ$  in such a way that  $\prec + \succ$  satisfy the identities (5)) and satisfies the extra axiom:

$$\Delta(x \prec y) = x_1 \prec y_1 \otimes x_2 \sqcup y_2.$$

Dually, the unshuffling coproduct  $\Delta = \Delta_{\prec} + \Delta_{\succ}$  can be defined recursively on  $T(V)$  by, for  $xX = xx_1 \dots x_n$ ,  $x, \dots, x_n \in V$ :

$$\Delta_{\prec}(x) := x \otimes 1, \quad \Delta_{\succ}(x) := 1 \otimes x;$$

$$\Delta_{\prec}(xX) = xX_1 \otimes X_2, \quad \Delta_{\succ}(xX) = X_1 \otimes xX_2,$$

where we used Sweedler's notation  $\Delta(X) = X_1 \otimes X_2$ . By duality with Schützenberger's axiomatic characterization of the shuffle product on the tensor algebra in terms of the identities satisfied by the half-shuffles [23] (see e.g. [11] for historical details), the half-unshufflings  $\Delta_{\prec}, \Delta_{\succ}$  satisfy the identities:

$$\Delta_{\prec} = T \circ \Delta_{\succ}, \quad (\Delta_{\prec} \otimes Id) \circ \Delta_{\prec} = (Id \otimes \Delta) \circ \Delta_{\prec},$$

where  $T$  stands for the switch map  $T(x \otimes y) = y \otimes x$ . These identities define the abstract notion of dual shuffle coalgebras (or coZinbiel coalgebras).

Using the shortcut  $\Delta_{\prec}(X) = X_1^{\prec} \otimes X_2^{\prec}$ , a dual shuffle bialgebra (or coZinbiel Hopf algebra, see e.g. [10] for further details) is a Hopf algebra equipped with a coassociative cocommutative coproduct  $\Delta = \Delta_{\prec} + \Delta_{\succ}$  satisfying the above identities and an associative product  $\cdot$  such that:

$$(6) \quad \Delta_{\prec}(X \cdot Y) = X_1^{\prec} \cdot Y_1 \otimes X_2^{\prec} \cdot Y_2.$$

A rigidity theorem due originally to Chapoton ([8, Thm. 1 and Prop. 12], see [11] for a direct and elementary proof) asserts that a shuffle bialgebra (resp., dually, a dual shuffle bialgebra) is canonically isomorphic to the tensor algebra equipped with the deconcatenation coproduct and the shuffle product (resp. the concatenation product and unshuffling coproduct). We are going to show how this structure theorem relates to Bernstein's ideas –in particular the one of “underlying Hopf algebra” of a cogroup in  $\mathcal{A}s$  introduced in [6] that, as we are going to show, is best understood using the notion of dual shuffle bialgebra.

Let  $H$  be a cocommutative cogroup in  $\mathcal{A}s$ . The structure map  $\phi : H \longrightarrow H * H$  gives rise to two “half-coproducts”  $\Delta_{\prec}, \Delta_{\succ}$  from  $H$  to  $H \otimes H$  defined as follows. Let  $h_1 \otimes \dots \otimes h_n \in (H * H)^{(n)}$ , we set:

$$\pi_1(h_1 \otimes \dots \otimes h_n) := 1_{h_1 \otimes \dots \otimes h_n \in (1, H^{\otimes n})} h_1 \cdot h_3 \cdot \dots \cdot h_{n-1} \otimes h_2 \cdot h_4 \cdot \dots \cdot h_n$$

$\pi_2(h_1 \otimes \dots \otimes h_n) := 1_{h_1 \otimes \dots \otimes h_n \in (2, H^{\otimes n})} h_2 \cdot h_4 \cdot \dots \cdot h_n \otimes h_1 \cdot h_3 \cdot \dots \cdot h_{n-1}$   
if  $n$  is even and else

$$\pi_1(h_1 \otimes \dots \otimes h_n) := 1_{h_1 \otimes \dots \otimes h_n \in (1, H^{\otimes n})} h_1 \cdot h_3 \cdot \dots \cdot h_n \otimes h_2 \cdot h_4 \cdot \dots \cdot h_{n-1},$$

$$\pi_2(h_1 \otimes \dots \otimes h_n) := 1_{h_1 \otimes \dots \otimes h_n \in (2, H^{\otimes n})} h_2 \cdot h_4 \cdot \dots \cdot h_{n-1} \otimes h_1 \cdot h_3 \cdot \dots \cdot h_n.$$

Then,

$$\Delta_{\prec}(h) := \pi_1 \circ \phi(h), \quad \Delta_{\succ}(h) := \pi_2 \circ \phi(h).$$

Maps  $\pi_i$ ,  $i = 1, 2, 3$  from  $H * H * H$  to  $H \otimes H \otimes H$  are defined similarly. That is, distinguishing notationally between the three copies of  $H$  by writing  $H * H * H = H_1 * H_2 * H_3$ ,  $\pi_1$  acts non trivially on  $h_1 \otimes \dots \otimes h_n \in H_1 * H_2 * H_3$  if and only if  $h_1 \in H_1$ , and so on.

**Proposition 5.** *The half-coproducts  $\Delta_{\prec}, \Delta_{\succ}$  together with the product define (functorially) on  $H$  the structure of a dual shuffle bialgebra.*

The identity  $\Delta_{\prec} = T \circ \Delta_{\succ}$  follows from the cocommutativity of  $\phi$ . The identity  $(\Delta_{\prec} \otimes Id) \circ \Delta_{\prec} = (Id \otimes \Delta) \circ \Delta_{\prec}$  follows by observing that both maps act as  $\pi_1 \circ \phi^{[3]}$  on  $H$ , where  $\phi^{[3]}$  is the iterated coproduct from  $H$  to  $H * H * H$ . The identity (6) follows from the fact that  $\phi$  is a morphism of algebras.

Berstein's notion of underlying Hopf algebra of a cogroup in  $\mathcal{A}s$  is obtained by composing this functor with the forgetful functor from dual shuffle bialgebras to classical bialgebras. Proposition 5 together with the following Theorem unravels why this notion of underlying Hopf algebra of a cogroup could prove in the end instrumental in Bernstein's work on cogroups in  $\mathcal{A}s$  (compare our approach to Bernstein's original one).

There also exists a functor (and an equivalence of categories) between cocommutative cogroups in  $\mathcal{A}s$  and shuffle bialgebras, whose explicit description is slightly more indirect. The existence of a functor from shuffle bialgebras to cocommutative cogroups in  $\mathcal{A}s$  follows from [11], where we showed that there is an explicit, natural isomorphism, from a shuffle bialgebra  $H$  to  $T(Prim(H))$ . The algebra of natural operations introduced in that article allows in particular the construction of a natural, explicit, map from  $H$  to  $Prim(H) \otimes H$  lifting the canonical isomorphisms  $Prim(H)^{\otimes n} = Prim(H) \otimes Prim(H)^{\otimes n-1}$  from which the cogroup structure on  $H$  can be defined recursively and explicitly.

The previous results can be gathered in the following theorem that generalizes [16, Sect. 5.2]:

**Theorem 6.** *The following categories are equivalent:*

- (1) *The category of graded connected vector spaces.*
- (2) *The subcategory of the category of graded connected algebras whose objects are the tensor algebras  $T(V)$  over graded vector spaces equipped with the concatenation product, with morphisms from  $T(V)$  to  $T(W)$  induced by linear maps from  $V$  to  $W$ .*
- (3) *The subcategory of the category of graded connected coalgebras whose objects are the tensor algebras  $T(V)$  over graded vector spaces equipped with the deconcatenation coproduct, with morphisms from  $T(V)$  to  $T(W)$  induced by linear maps from  $V$  to  $W$ .*
- (4) *The category of cocommutative cogroups in  $\mathcal{A}s$ .*

- (5) *The category of graded connected infinitesimal bialgebras.*
- (6) *The category of graded connected shuffle bialgebras.*
- (7) *The category of graded connected dual shuffle bialgebras.*

The last four are actually isomorphic, that is could be related by inverse functors acting as the identity on objects. We have constructed some of them explicitly and will develop further these ideas in a forthcoming article.

The equivalence of the first three items is straightforward, but we include it to make clear that the points of view of free algebras and cofree coalgebras lead to two different approaches to the characterization of the  $T(V)$ s.

The equivalence of the first four items is Bernstein's structure theorem for cocommutative cogroups in  $\mathcal{A}s$ . The relations between  $\phi$  and  $\Delta$  in equation (2) make explicit the functorial equivalence between (4) and (5). The equivalence of (1) and (5) was first proven directly in [17]. The functor of primitive elements and the tensor algebra functor underly the equivalence between (4,5,6) and (1). The equivalence of (1) and (6,7) follows from Chapoton's structure theorem for shuffle bialgebras and the dual statement for dual shuffle bialgebras [8]. The functor describing the equivalence between (4) and (7) is the object of Proposition 5. The functors describing the equivalence between (4) and (6) can be constructed explicitly following the methods explained before the statement of the Theorem.

**Corollary 7.** *The following statements are equivalent (as usual all objects are graded, connected):*

- (1)  *$H$  is a Hopf algebra, cofree over the space of its primitive elements  $V = \text{Prim}(H)$ .*
- (2)  *$H$  is the  $B_\infty$ -enveloping algebra of a  $B_\infty$ -algebra  $V$ .*
- (3)  *$H$  is a Hopf algebra and can be equipped with the structure of a cocommutative cogroup in  $\mathcal{A}s$  such that the coproduct  $\Delta$  of  $H$  is the one associated to the structure map  $\phi : H \rightarrow H * H$ .*
- (4)  *$H$  is a Hopf algebra and can be equipped with the structure of an infinitesimal bialgebra whose coproduct is the coproduct of  $H$  (equivalently, in the language of [17],  $H$  can be equipped with the structure of a 2-associative bialgebra extending its Hopf algebra structure).*
- (5)  *$H$  is a Hopf algebra and can be equipped with the structure of a shuffle bialgebra whose coproduct is the coproduct of  $H$ .*

## 5. $B_\infty$ -ALGEBRAS AND FINITE SPACES

**Notations.** Let  $X$  be a finite set, and  $\mathcal{T}$  be a topology on  $X$ . For any  $Y \subseteq X$ , we denote by  $\mathcal{T}|_Y$  the topology induced by  $\mathcal{T}$  on  $Y$ , that is to say:

$$\mathcal{T}|_Y = \{O \cap Y \mid O \in \mathcal{T}\}.$$

**Definition 8.** Let  $\mathcal{T} \in \mathbf{T}_n$ ,  $n \geq 1$ . For  $\overline{\mathcal{T}} \in \mathbf{F}_n$ , the equivalence class of  $\mathcal{T}$  in  $\mathbf{F}$ , we put:

$$\Delta(\overline{\mathcal{T}}) := \sum_{O \in \mathcal{T}} \overline{\mathcal{T}|_{[n] \setminus O}} \otimes \overline{\mathcal{T}|_O}.$$

We let the reader check that this definition does not depend of the choice of a representative of  $\overline{\mathcal{T}}$  in  $\mathbf{T}$ . The coproduct extends linearly to  $\mathbf{F}$ , the linear span of finite spaces.

- Theorem 9.** (1)  $(\mathbf{F}, \cdot, \Delta)$  is a graded connected commutative Hopf algebra.  
 (2)  $(\mathbf{F}, \succ, \Delta)$  is a graded connected infinitesimal bialgebra.  
 (3)  $\mathbf{F}$  is the  $B_\infty$ -enveloping algebra of a  $B_\infty$ -algebra; more precisely it is a cofree graded connected commutative Hopf algebra. It can be equipped with the structure of a cocommutative cogroup in  $\mathcal{A}s$ , of a shuffle bialgebra or of a dual shuffle bialgebra.

*Proof.* The last assertion follows from the previous ones together with Corollary 7.

Let  $\mathcal{T} \in \mathbf{T}_n$ ,  $n > 0$ . The coassociativity of  $\Delta$  follows from the observations that:

- if  $O$  is open in  $\mathcal{T}$ , then the open sets of  $O$  are the open sets of  $\mathcal{T}$  contained in  $O$ ,
- if  $O \in \mathcal{T}$  and  $O' \in \mathcal{T}_{[n] \setminus O}$ , then  $O \sqcup O'$  is an open set of  $\mathcal{T}$ ,
- if  $O_1 \subseteq O_2$  are open sets of  $\mathcal{T}$ , then  $O_2 \setminus O_1 \in \mathcal{T}_{[n] \setminus O_1}$ .

We get then:

$$\begin{aligned} (\Delta \otimes Id) \circ \Delta(\overline{\mathcal{T}}) &= \sum_{O \in \mathcal{T}, O' \in \mathcal{T}_{[n] \setminus O}} \overline{(\mathcal{T}_{[n] \setminus O})_{|([n] \setminus O) \setminus O'}} \otimes \overline{(\mathcal{T}_{[n] \setminus O})_{|O'}} \otimes \overline{\mathcal{T}_O} \\ &= \sum_{O \in \mathcal{T}, O' \in \mathcal{T}_{[n] \setminus O}} \overline{\mathcal{T}_{[n] \setminus (O \sqcup O')}} \otimes \overline{\mathcal{T}_{O'}} \otimes \overline{\mathcal{T}_O}. \end{aligned}$$

Putting  $O_1 = O$  and  $O_2 = O \sqcup O'$ :

$$(\Delta \otimes Id) \circ \Delta(\overline{\mathcal{T}}) = \sum_{O_1 \subseteq O_2 \in \mathcal{T}} \overline{\mathcal{T}_{[n] \setminus O_2}} \otimes \overline{\mathcal{T}_{O_2 \setminus O_1}} \otimes \overline{\mathcal{T}_{O_1}} = (Id \otimes \Delta) \circ \Delta(\overline{\mathcal{T}}).$$

This proves that  $\Delta$  is coassociative. It is obviously homogeneous of degree 0. Moreover,  $\Delta(1) = 1 \otimes 1$  and for any  $\mathcal{T} \in \mathbf{T}_n$ ,  $n \geq 1$ :

$$\Delta(\overline{\mathcal{T}}) = \overline{\mathcal{T}} \otimes 1 + 1 \otimes \overline{\mathcal{T}} + \sum_{\emptyset \subsetneq O \subsetneq [n]} \overline{\mathcal{T}_{[n] \setminus O}} \otimes \overline{\mathcal{T}_O}.$$

So  $\Delta$  has a counit.

Let  $\mathcal{T} \in \mathbf{T}_n$ ,  $\mathcal{T}' \in \mathbf{T}_{n'}$ ,  $n, n' \geq 0$ . By definition of  $\mathcal{T} \cdot \mathcal{T}'$ :

$$\begin{aligned} \Delta(\overline{\mathcal{T} \cdot \mathcal{T}'}) &= \sum_{O \in \mathcal{T}, O' \in \mathcal{T}'} \overline{(\mathcal{T} \cdot \mathcal{T}')_{|[n+n'] \setminus O \cdot O'}} \otimes \overline{(\mathcal{T} \cdot \mathcal{T}')_{|O \cdot O'}} \\ &= \sum_{O \in \mathcal{T}, O' \in \mathcal{T}'} \overline{\mathcal{T}_{[n] \setminus O} \cdot \mathcal{T}'_{[n'] \setminus O'}} \otimes \overline{\mathcal{T}_O \cdot \mathcal{T}'_{O'}} \\ &= \sum_{O \in \mathcal{T}, O' \in \mathcal{T}'} (\overline{\mathcal{T}_{[n] \setminus O}} \otimes \overline{\mathcal{T}_O}) \cdot (\overline{\mathcal{T}'_{[n'] \setminus O'}} \otimes \overline{\mathcal{T}'_{O'}}) \\ &= \Delta(\overline{\mathcal{T}}) \cdot \Delta(\overline{\mathcal{T}'}). \end{aligned}$$

Hence,  $(\mathbf{F}, \cdot, \Delta)$  is a graded connected commutative Hopf algebra.

By definition of  $\mathcal{T} \succ \mathcal{T}'$ :

$$\begin{aligned}
\Delta(\overline{\mathcal{T}} \succ \overline{\mathcal{T}}') &= \sum_{O \in \mathcal{T}, O \neq \emptyset} \overline{(\mathcal{T} \succ \mathcal{T}')_{|[n+n'] \setminus (O \succ [n'])}} \otimes \overline{(\mathcal{T} \succ \mathcal{T}')_{|O \succ [n']}} \\
&\quad + \sum_{O' \in \mathcal{T}', O' \neq [n']} \overline{(\mathcal{T} \succ \mathcal{T}')_{|[n+n'] \setminus O'(+n)}} \otimes \overline{(\mathcal{T} \succ \mathcal{T}')_{|O'(+n)}} \\
&\quad + \overline{(\mathcal{T} \succ \mathcal{T}')_{|[n+n'] \setminus [n'](+n)}} \otimes \overline{(\mathcal{T} \succ \mathcal{T}')_{|[n'](+n)}} \\
&= \sum_{O \in \mathcal{T}, O \neq \emptyset} \overline{\mathcal{T}_{|[n] \setminus O}} \otimes \overline{\mathcal{T}_{|O}} \succ \overline{\mathcal{T}'} \\
&\quad + \sum_{O' \in \mathcal{T}', O' \neq [n']} \overline{\mathcal{T}} \succ \overline{\mathcal{T}'_{|[n'] \setminus O'}} \otimes \overline{\mathcal{T}'_{|O'}} + \overline{\mathcal{T}} \otimes \overline{\mathcal{T}'} \\
&= \sum_{O \in \mathcal{T}, O \neq \emptyset} (\overline{\mathcal{T}_{|[n] \setminus O}} \otimes \overline{\mathcal{T}_{|O}}) \succ (1 \otimes \overline{\mathcal{T}'}) \\
&\quad + \sum_{O' \in \mathcal{T}', O' \neq [n']} (\overline{\mathcal{T}} \otimes 1) \succ (\overline{\mathcal{T}'_{|[n'] \setminus O'}} \otimes \overline{\mathcal{T}'_{|O'}}) + \overline{\mathcal{T}} \otimes \overline{\mathcal{T}'} \\
&= (\Delta(\overline{\mathcal{T}}) - \overline{\mathcal{T}} \otimes 1) \succ (1 \otimes \overline{\mathcal{T}'}) + (\overline{\mathcal{T}} \otimes 1) \succ (\Delta(\overline{\mathcal{T}}) - 1 \otimes \overline{\mathcal{T}'}) + \overline{\mathcal{T}} \otimes \overline{\mathcal{T}'} \\
&= \Delta(\overline{\mathcal{T}}) \succ (1 \otimes \overline{\mathcal{T}'}) + (\overline{\mathcal{T}} \otimes 1) \succ \Delta(\overline{\mathcal{T}}) - \overline{\mathcal{T}} \otimes \overline{\mathcal{T}'}.
\end{aligned}$$

Hence,  $(\mathbf{F}, \succ, \Delta)$  is an infinitesimal Hopf algebra.  $\square$

#### REFERENCES

1. P. Alexandroff, *Diskrete Räume*, Rec. Math. Moscou **2** (1937), 501–519.
2. J. A. Barmak, *Algebraic topology of finite topological spaces and applications*. PhD Thesis, Universidad de Buenos Aires, 2009.
3. J. A. Barmak and E. G. Minian, *Minimal finite models*, Journal of Homotopy and Related Structures **2** (2007), no. 1, 127–140.
4. J. A. Barmak and E. G. Minian, *Strong homotopy types, nerves and collapses*, Discrete Comput. Geom. **47** (2012), no. 2, 301–328.
5. H. J. Baues, *The double bar and cobar constructions*, Compositio Math. **43** (1981), no. 3, 331–341.
6. I. Bernstein, *On co-groups in the category of graded algebras.*, Trans. Am. Math. Soc. **115** (1965), 257–259.
7. R. Bott and H. Samelson, *On the Pontryagin product in spaces of paths*, Comment. Math. Helv. **27** (1953), 320–337.
8. F. Chapoton, *Un théorème de cartier–milnor–moore–quillen pour les bigèbres dendri-formes et les algèbres braces*, Journal of Pure and Applied Algebra **168** (2002), no. 1, 1–18.
9. M. Ern  and K. Stege, *Counting finite posets and topologies*, Order **8** (1991), no. 3, 247–265.
10. F. Fisher, *Cozinbiel hopf algebras in combinatorics*, Ph.D. thesis, The George Washington University, 2010.
11. L. Foissy and F. Patras, *Natural endomorphisms of shuffle algebras*, International Journal of Algebra and Computation **23** (2013), no. 4, 989–1009.
12. B. Fresse, *Alg bre des descentes et cogroupes dans les alg bres sur une op rade*, Bulletin de la Soci t  Math matique de France **126** (1998), no. 3, 107–134.
13. ———, *Cogroups in algebras over an operad are free algebras*, Commentarii Mathematici Helvetici **73** (1998), no. 4, 637–676.

14. E. Getzler and J. D. S. Jones, *Operads, homotopy algebra and iterated integrals for double loop spaces*, arXiv preprint hep-th/9403055 (1994).
15. M. Hazewinkel, N. M. Gubareni, and V. V. Kiričenko, *Algebras, rings, and modules: Lie algebras and hopf algebras*, no. 168, AMS Bookstore, 2010.
16. M. Livernet, *A rigidity theorem for pre-Lie algebras.*, J. Pure Appl. Algebra **207** (2006), no. 1, 1–18.
17. J.-L. Loday and M. Ronco, *On the structure of cofree Hopf algebras*, J. Reine Angew. Math. **592** (2006), 123–155.
18. M.C. McCord, *Singular homology groups and homotopy groups of finite topological spaces.*, Duke Math. J. **33** (1966), 465–474.
19. ———, *Homotopy type comparison of a space with complexes associated with its open covers.*, Proc. Am. Math. Soc. **18** (1967), 705–708.
20. J.-M. Oudom, *Théoreme de leray dans la catégorie des algebres sur une opérade*, Comptes Rendus de l'Académie des Sciences-Series I **329** (1999), no. 2, 101–106.
21. F. Patras, *A leray theorem for the generalization to operads of hopf algebras with divided powers*, Journal of Algebra **218** (1999), no. 2, 528–542.
22. C. Reutenauer, *Free lie algebras*, London Mathematical Society Monographs, New Series 7, Oxford University Press, 1993.
23. M. P. Schützenberger, *Sur une propriété combinatoire des algebres de lie libres pouvant être utilisée dans un problème de mathématiques appliquées*, Séminaire Dubreil. Algèbre et théorie des nombres **12** (1958), no. 1, 1–23.
24. H. Sharp, *Cardinality of finite topologies*, Journal of Combinatorial Theory **5** (1968), no. 1, 82–86.
25. N. J. A. Sloane, *On-line encyclopedia of integer sequences*, <http://oeis.org/>.
26. R. P Stanley, *On the number of open sets of finite topologies*, Journal of Combinatorial Theory, Series A **10** (1971), no. 1, 74–79.
27. R. E. Stong, *Finite topological spaces*, Trans. Amer. Math. Soc. **123** (1966), 325–340.

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